A1)

$$\left(\begin{array}{ccc|c}1&1&1&10\\2&-1&3&4\\1&0&2&20\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}1&1&1&10\\2&-1&3&4\\0&-1&1&10\end{array}\right)$$
$$\rightarrow\left(\begin{array}{ccc|c}1&1&1&10\\0&-3&1&-16\\0&-1&1&10\end{array}\right)$$
$$\rightarrow\left(\begin{array}{ccc|c}1&1&1&10\\0&-3&1&-16\\0&0&2&46\end{array}\right)$$
$$2z=46\Rightarrow z=23$$
$$-3y+z=-16\Rightarrow y=13$$
$$x+y+z=10\Rightarrow x=-26$$

A2)

a)

$$f(x)=(2+x)\tan^{-1}\sqrt{x-1}$$
$$f'(x)=\tan^{-1}\sqrt{x-1}+(2+x)\frac{1}{1+x-1}\times\frac{1}{2}(x-1)^{-\frac{1}{2}}$$
$$=\tan^{-1}\sqrt{x-1}+\frac{2+x}{2x\sqrt{x-1}}$$

b)

$$g(x)=e^{\cot 2x}$$
$$g'(x)=e^{\cot 2x}\frac{d(\cot 2x)}{dx}=-2e^{\cot 2x}\csc^2 2x$$

A3)

$$\int_0^{\frac{\pi}{4}}2x\sin 4x\,dx=\left[-\frac{1}{2}x\cos 4x+\frac{1}{2}\int\cos 4x\,dx\right]_0^{\frac{\pi}{4}}$$
$$=\left[-\frac{1}{2}x\cos 4x+\frac{1}{8}\sin 4x\right]_0^{\frac{\pi}{4}}$$
$$=\left[\frac{\pi}{8}+0+0-0\right]=\frac{\pi}{8}$$

A4)

Base case:
Let $n=1$, then,

$$\text{LHS}=\sum_{k=1}^{n}(3k-1)=3n-1=3-1=2$$
$$\text{RHS}=\frac{1}{2}n(3n+1)=\frac{1}{2}(3+1)=\frac{1}{2}\times 4=2=\text{LHS}$$

Therefore it is true for the base case $n=1$

Assume it is true for $n=m$, $m\geq 1$, $m\in\mathbb{Z}$:

$$\sum_{k=1}^{n}(3n-1)=\sum_{k=1}^{m}(3m-1)=\frac{1}{2}m(3m+1)$$

Then for $n=m+1$:

$$\sum_{k=1}^{n}(3k-1)=\sum_{k=1}^{m+1}(3k-1)=3\sum_{k=1}^{m+1}k-\sum_{k=1}^{m+1}1$$
$$=\frac{3}{2}(m+1)(m+2)-(m+1)$$
$$=\frac{1}{2}(m+1)\left[3(m+2)-2\right]$$
$$=\frac{1}{2}(m+1)(3m+4)=\frac{1}{2}(m+1)\left[3(m+1)+1\right]$$
$$=\frac{1}{2}n(3n+1)$$

Since the conjecture is true for $n=1$ and also true for $n=m+1$ when assuming true for $n=m\in\mathbb{Z}$, $m\geq 1$, by induction it is true for all $n\in\mathbb{Z}$, $n\geq 1$.

A5)

a)

$$\frac{x}{x^2-1}=\frac{x}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}$$
$$x=A(x-1)+B(x+1)$$

$$x=1:\quad 1=2B\Rightarrow B=\frac{1}{2}$$
$$x=-1:\quad -1=-2A\Rightarrow A=\frac{1}{2}$$

$$\frac{x}{x^2-1}=\frac{1}{2(x+1)}+\frac{1}{2(x-1)}$$

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A5b)

$$\int \frac{x^3}{x^2-1}dx = \int x^2 \frac{x}{x^2-1}dx$$
$$= \int \left[\frac{x^2}{2(x+1)} + \frac{x^2}{2(x-1)} \right] dx$$
$$= \frac{1}{2}\int \left[\frac{x^2-1}{x+1} + \frac{1}{x+1} + \frac{x^2-1}{x-1} + \frac{1}{x-1} \right] dx$$
$$= \frac{1}{2}\int \left[x-1+\frac{1}{x+1} + x+1+\frac{1}{x-1} \right] dx$$
$$= \frac{1}{2}\left[x^2 + \ln(x+1) + \ln(x-1) \right] + C$$
$$= \frac{1}{2}\left[x^2 + \ln(x^2-1) \right] + C$$

A6)

$$\left(x^2 - \frac{2}{x}\right)^4 = \sum_{k=0}^{4} \binom{4}{k} (x^2)^{4-k} \left(-\frac{2}{x}\right)^k$$
$$= \binom{4}{0}(x^2)^4 + \binom{4}{1}(x^2)^3\left(-\frac{2}{x}\right) + \binom{4}{2}(x^2)^2\left(-\frac{2}{x}\right)^2$$
$$+ \binom{4}{3}(x^2)\left(-\frac{2}{x}\right)^3 + \binom{4}{4}\left(-\frac{2}{x}\right)^4$$
$$= x^8 - 8x^5 + 24x^2 - \frac{32}{x} + \frac{16}{x^4}$$

A7)

a)

$$xy + y^2 = 2$$
$$y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(x+2y) = -y$$
$$\frac{dy}{dx} = \frac{-y}{x+2y}$$

A7b)

$$x = 1,\ y = 1:$$
$$m = \frac{dy}{dx} = \frac{-1}{1+2} = \frac{-1}{3}$$
$$y - 1 = \frac{-1}{3}(x-1)$$
$$3y - 3 = 1 - x$$
$$3y = 4 - x$$

A8)

a)

$$f(x) = \frac{x^2+6x+12}{x+2}$$

$$\begin{array}{r} x+\ \ 4 \\ (x+2)\overline{)\,x^2+6x+12} \\ \underline{x^2+2x} \\ 4x+12 \\ \underline{4x+\ \ 8} \\ 4 \end{array}$$

$$\Rightarrow f(x) = x + 4 + \frac{4}{x+2},\ a = 1,\ b = 4$$

b)

$$x = -2\ (\text{vertical})$$
$$y = x + 4\ (\text{non-vertical})$$

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A8c)

$f\left(x\right)=x+4+\frac{4}{x+2}$

For S.P.s:

$f'\left(x\right)=1-\frac{4}{\left(x+2\right)^2}=0$

$\left(x+2\right)^2=4\Rightarrow x+2=\pm 2\Rightarrow x=0,\ -4$

Two solutions so two stationary points.

$f''\left(x\right)=\frac{8}{\left(x+2\right)^3}$

$x=0:$

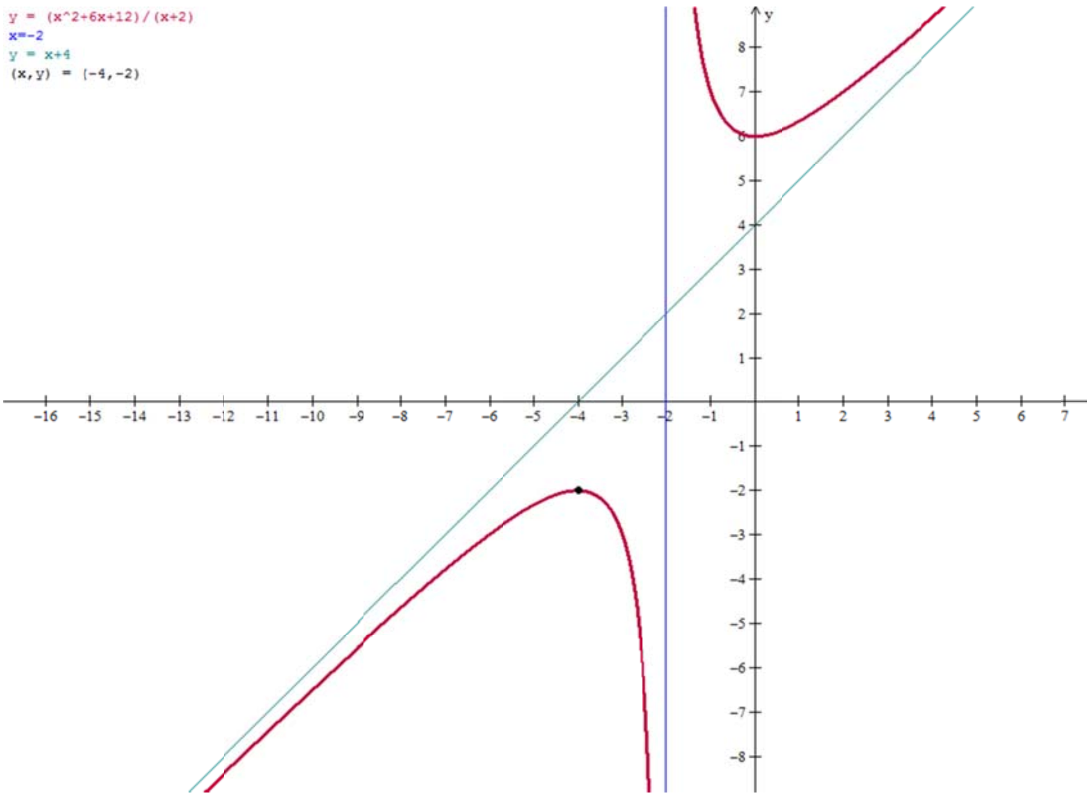
$f\left(x\right)=4+\frac{4}{2}=6$

$f''\left(x\right)=\frac{8}{\left(0+2\right)^3}=1>0\Rightarrow$ Minimum at $\left(0,6\right)$

$x=-4:$

$f\left(x\right)=-4+4+\frac{4}{2-4}=-2$

$f''\left(x\right)=\frac{8}{\left(2-4\right)^3}=-1<0\Rightarrow$ Maximum at $\left(-4,-2\right)$



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A8d)

$-2 < k < 6$

A9)

a)

$\mathrm{Im} = 0,\ \mathrm{Re} = -1 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$

b)

$z^3 + 1 = 0$

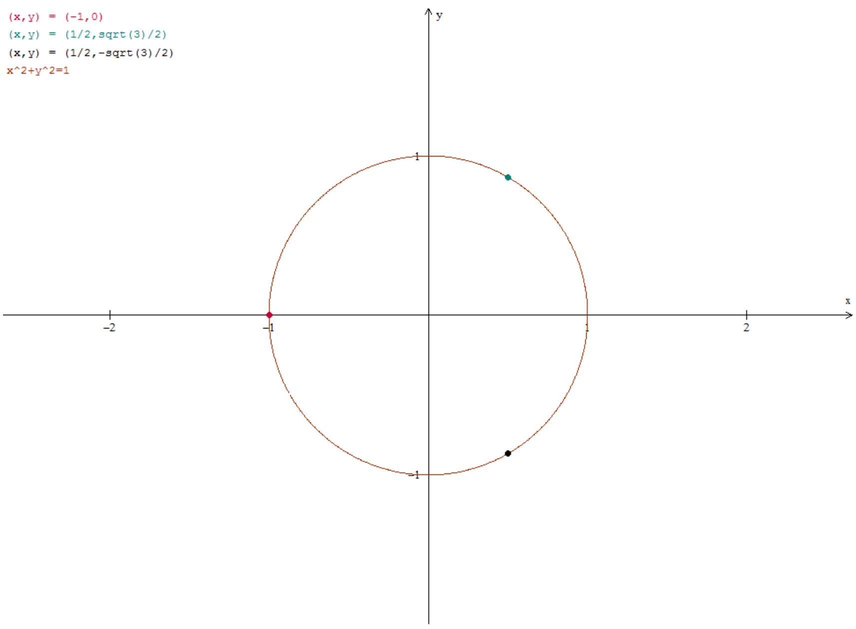
$$\begin{aligned}
z^3 &= -1 \\
z &= \cos\pi + i\sin\pi \\
z &= \cos 3\pi + i\sin 3\pi \\
z &= \cos 5\pi + i\sin 5\pi
\end{aligned}$$

$$\begin{aligned}
z &= \left(\cos\pi + i\sin\pi\right)^{\frac{1}{3}} \\
&= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{i\sqrt{3}}{2}
\end{aligned}$$

$$\begin{aligned}
z &= \left(\cos 3\pi + i\sin 3\pi\right)^{\frac{1}{3}} \\
&= \cos\pi + i\sin\pi = -1
\end{aligned}$$

$$\begin{aligned}
z &= \left(\cos 5\pi + i\sin 5\pi\right)^{\frac{1}{3}} \\
&= \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

The points are placed on the unit circle centred on the origin and divide the circumference in equal distances.



A10)

a)

$$\frac{dM}{dt} = kM$$

$$\frac{dM}{M} = kdt$$

$$\int \frac{dM}{M} = \int kdt$$

$$\ln M = kt + C$$

$$M = Ae^{kt}, \quad A = e^{C}$$

$$M_0 = Ae^{0} = A$$

$$M = M_0 e^{kt}$$

b)

$$\frac{1}{2}M_0 = M_0 e^{30k}$$

$$e^{30k} = \frac{1}{2}$$

$$30k = \ln\frac{1}{2}$$

$$k = \frac{1}{30}\ln\frac{1}{2} \approx -0.231$$

c)

$$M = M_0 e^{35\times\frac{1}{30}\ln\frac{1}{2}} = M_0 e^{\frac{7}{6}\ln\frac{1}{2}}$$

$$= M_0 \frac{1}{2^{7/6}} \approx 0.445 M_0$$

44.5%

d)

$$\frac{1}{4}M_0 = M_0 e^{kt}$$

$$e^{kt} = \frac{1}{4}$$

$$kt = \ln\frac{1}{4} = \frac{t}{30}\ln\frac{1}{2}$$

$$t = 30\frac{\ln 0.25}{\ln 0.5} = 2\times 30 = 60 \text{ days}$$

The claim is justified.

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B1)

$$
\begin{aligned}
149 &= 1\cdot 139 + 10 \\
139 &= 13\cdot 10 + 9 \\
10 &= 1\cdot 9 + 1 \\
9 &= 9\cdot 1 + 0
\end{aligned}
$$

$$\gcd(139,149) = 1$$

$$1 = 10 - 1\cdot 9$$

$$9 = 139 - 13\cdot 10$$
$$\Rightarrow 1 = 10 - 1\cdot(139 - 13\cdot 10) = 14\cdot 10 - 1\cdot 139$$

$$10 = 149 - 1\cdot 139$$
$$\Rightarrow 1 = 14\cdot(149 - 1\cdot 139) - 1\cdot 139 = 14\cdot 149 - 15\cdot 139$$

$$x = 14,\ y = -15$$

B2)

$$\frac{dy}{dx} + \frac{y}{x} = x$$

$$I(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$x\frac{dy}{dx} + y = x^2 = \frac{d(Iy)}{dx}$$

$$Iy = xy = \int x^2 dx = \frac{x^3}{3} + C$$

$$y = \frac{x^2}{3} + \frac{C}{x}$$

B3)

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\mathbf{I}$$

i, ii)

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\mathbf{I}$$

$$\mathbf{AB} = 2\mathbf{I} \Rightarrow \mathbf{B} = 2\mathbf{A}^{-1} \Rightarrow \mathbf{A}^{-1} = \frac{1}{2}\mathbf{B}$$

$$\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}$$

$$\mathbf{AB} = 2\mathbf{I} \Rightarrow \mathbf{A}^2\mathbf{B} = 2\mathbf{A}$$

$$\mathbf{A}^2\mathbf{B} = 2\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 6 \\ 2 & -2 & -2 \end{pmatrix}$$

B4)

$$x = 0:$$

$$f(x) = (2+x)\ln(2+x) = 2\ln 2$$

$$f'(x) = \ln(2+x) + \frac{2+x}{2+x} = 1 + \ln 2$$

$$f''(x) = \frac{1}{2+x} = \frac{1}{2}$$

$$f'''(x) = -\frac{1}{(2+x)^2} = -\frac{1}{4}$$

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(x)\frac{x^n}{n!}$$
$$\approx 2\ln 2 + x(1+\ln 2) + \frac{x^2}{4} - \frac{x^3}{24} + \ldots$$

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B5)

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=6x-1$$

$$m^2+2m-3=0$$
$$(m+3)(m-1)=0$$
$$m=1,\ \ -3$$
$$y_c=Ae^x+Be^{-3x}$$

$$y_p=ax+b$$
$$\frac{dy_p}{dx}=a$$
$$\frac{d^2y_p}{dx^2}=0$$
$$\frac{d^2y_p}{dx^2}+2\frac{dy_p}{dx}-3y_p=2a-3ax-3b=6x-1$$
$$\Rightarrow -3a=6\Rightarrow a=-2$$
$$\Rightarrow 2a-3b=-1\Rightarrow -3b=3\Rightarrow b=-1$$
$$y_p=-2x-1$$

$$y=y_c+y_p=Ae^x+Be^{-3x}-2x-1$$

B6)

ai)

$L_1: x=8-2t,\ y=-4+2t,\ z=3+t$

$L_2: \frac{x}{-2}=\frac{y+2}{-1}=\frac{z-9}{2}$

The lines intersect if and only if:

$$\frac{8-2t}{-2}=\frac{-4+2t+2}{-1}=\frac{3+t-9}{2}=v$$
$$=t-4=2-2t=\frac{t}{2}-3$$
$$\Rightarrow t=2,\ v=-2$$

A consistent solution for t exists, so the lines intersect.

$$x=8-2t=4$$
$$y=-4+2t=0$$
$$z=3+t=5$$
$$(4,0,5)$$

aii)

$$\mathbf{d}_1=\begin{pmatrix}-2\\2\\1\end{pmatrix},\ \mathbf{d}_2=\begin{pmatrix}-2\\-1\\2\end{pmatrix}$$
$$\mathbf{d}_1\cdot\mathbf{d}_2=-4-2+2=-4$$
$$|\mathbf{d}_1|=\sqrt{9}=3$$
$$|\mathbf{d}_2|=3$$
$$\cos\theta=\frac{\mathbf{d}_1\cdot\mathbf{d}_2}{|\mathbf{d}_1||\mathbf{d}_2|}=\frac{-4}{9}$$
$$\Rightarrow \theta=\cos^{-1}\left(\frac{-4}{9}\right)=\cos^{-1}\left(\frac{4}{9}\right)$$

bi)

$$\mathbf{n}=\begin{pmatrix}-2\\-1\\2\end{pmatrix},\ \mathbf{a}=\begin{pmatrix}1\\-4\\2\end{pmatrix}$$
$$\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$$
$$-2x-y+2z=-2+4+4=6$$

bii)

$$-2x-y+2z=6$$
$$-2(8-2t)-(-4+2t)+2(3+t)=6$$
$$-16+4t+4-2t+6+2t=6$$
$$4t-6=6\Rightarrow t=3$$

$$x=8-2t=8-6=2$$
$$y=-4+2t=-4+6=2$$
$$z=3+t=3+3=6$$
$$(2,2,6)$$